J. DIFFERENTIAL GEOMETRY
65 (2003) 1-17

HOLOMORPHIC LINE BUNDLES ON THE LOOP SPACE OF THE RIEMANN SPHERE

NING ZHANG

Abstract

The loop space $L\mathbb{P}_1$ of the Riemann sphere consisting of all C^k or Sobolev $W^{k,p}$ maps $S^1 \to \mathbb{P}_1$ is an infinite dimensional complex manifold. The loop group $LPGL(2,\mathbb{C})$ acts on $L\mathbb{P}_1$. We prove that the group of $LPGL(2,\mathbb{C})$-invariant holomorphic line bundles on $L\mathbb{P}_1$ is isomorphic to an infinite dimensional Lie group. Further, we prove that the space of holomorphic sections of any such line bundle is finite dimensional, and compute the dimension for a generic bundle.

1. Introduction

Let M be a finite dimensional complex manifold. Its loop space LM with a specified regularity, for example C^k $(1 \le k \le \infty)$ or $W^{k,p}$ $(1 \le k < \infty, 1 \le p < \infty)$, consists of all maps of the circle S^1 into M with the given regularity. LM is an infinite dimensional complex manifold. This paper studies holomorphic line bundles on the loop space $L\mathbb{P}_1$ of the Riemann sphere.

A direct motivation comes from [9], where Millson and Zombro conjecture that there exists a $PGL(2,\mathbb{C})$-equivariant embedding of $L\mathbb{P}_1$ into a projectivized Banach/Fréchet space. The conjecture arises in connection with extending Mumford's geometric invariant theory to an infinite dimensional setting. Another indirect motivation comes from [11], where Witten suggests to study the geometry and analysis of real and complex manifolds through their loop spaces. In finite dimensions it is a problem of fundamental importance to identify the Picard group of holomorphic line bundles on

This research was partially supported by NSF grant DMS-0203072.
Received 07/16/2002.

a complex manifold and the space of holomorphic sections of these bundles. Here we address this problem for a class of holomorphic line bundles on the first interesting loop space: $L\mathbb{P}_1$, and in particular make some progress toward answering the conjecture by Millson and Zombro.

The following are the main results of this paper:

If a group G acts on a set V, let V^G denote the G-fixed subset of V. The loop space of a finite dimensional complex Lie group is a complex Lie group under pointwise group operation (loop group). Let $\mathrm{Pic}(L\mathbb{P}_1)$ be the Picard group of $L\mathbb{P}_1$. The group $PGL(2,\mathbb{C})$ acts on $\mathbb{P}_1$, so the loop group $LPGL(2,\mathbb{C})$ acts on $L\mathbb{P}_1$ and on $\mathrm{Pic}(L\mathbb{P}_1)$. Let $L\mathbb{C}^*$ be the loop group of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$ be the group of holomorphic homomorphisms from $L\mathbb{C}^*$ to $\mathbb{C}^*$.

Theorem 1.1. $\mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})} \cong \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$ *as groups.*

Note that $\mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$ is a $\mathbb{Z}$-module of infinite rank, while the group of topological isomorphism classes of line bundles on $L\mathbb{P}_1$ is isomorphic to $\mathbb{Z}$ (cf. [8]).

Evaluation of loops in $\mathbb{P}_1$ at $t \in S^1$ gives rise to a holomorphic map $E_t : L\mathbb{P}_1 \to \mathbb{P}_1$. Let $\Lambda_t \in \mathrm{Pic}(L\mathbb{P}_1)$ be the pull back of the hyperplane bundle on $\mathbb{P}_1$ by E_t.

Theorem 1.2. *Let* $\Lambda \in \mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$.

(1) *If* $\Lambda \cong \Lambda_{t_1}^{n_1} \otimes \cdots \otimes \Lambda_{t_r}^{n_r}$, *where* $n_i \geq 0$ *and* $t_i \neq t_j$ *for* $i \neq j$, *then* $(n_1 + 1) \cdots (n_r + 1) \leq \dim H^0(L\mathbb{P}_1, \Lambda) < \infty$.

(2) *Otherwise* $H^0(L\mathbb{P}_1, \Lambda) = 0$.

Therefore the sections of no $LPGL(2,\mathbb{C})$-invariant holomorphic line bundle will give rise to a projective embedding of $L\mathbb{P}_1$.

The isomorphism in Theorem 1.1 is gotten by an explicit construction in Section 2. In Section 3 we prove Theorem 1.2(2). In Section 4 we study the space of holomorphic sections or the zero order Dolbeault cohomology group of line bundles defined in Theorem 1.2(1), and in particular prove Theorem 1.2(1). Any such line bundle obviously has holomorphic sections: products of pulled back sections by the evaluation maps. We will show that for a generic bundle of this type these are all sections. Yet there are bundles which have other sections as well; interestingly, in this case $\dim H^0(L\mathbb{P}_1, \Lambda_\varphi)$ depends on the regularity of the loops.

It is natural to ask whether $\mathrm{Pic}(L\mathbb{P}_1)^{PGL(2,\mathbb{C})} = \mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$. If so, then the conjecture made by Millson and Zombro is answered in the negative.

The author would like to thank Professor László Lempert for many helpful conversations and his continuous encouragement. Thanks are also due to Professor John J. Millson and Professor James McClure for valuable communications on some issues related to this paper.

2. Identification of $\mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$

We fix a regularity class $\mathcal{F}$ among C^k $(1 \le k \le \infty)$ respectively $W^{k,p}$ $(1 \le k < \infty, 1 \le p < \infty)$. In this paper we write LM (L_kM resp. $L_{k,p}M$) to denote the $\mathcal{F}$ (C^k resp. $W^{k,p}$) loop space of a manifold M. Let M and N be finite dimensional complex manifolds and $\phi : M \to N$ be a holomorphic map. Define $L\phi : LM \ni x \mapsto \phi \circ x \in LN$. Then LM and LN are infinite dimensional complex manifolds locally biholomorphic to open subsets of complex Banach (Fréchet when $\mathcal{F} = C^\infty$) spaces, and $L\phi$ is holomorphic. Thus L is a functor from the category of finite dimensional complex manifolds to the category of all complex manifolds. Let $t \in S^1$. The evaluation map $E_t = E_t^{LM} : LM \ni x \mapsto x(t) \in M$ is holomorphic. See Section 2 of [6].

We call constant maps $S^1 \to M$ point loops in M. They form a submanifold of LM, which we identify with M.

Next we define a map $\mathcal{L} : \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \to \mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$. We will show that $\mathcal{L}$ is an isomorphism of groups, which will then prove Theorem 1.1.

In Section 6 of [9] Millson and Zombro construct a holomorphic line bundle on $L\mathbb{P}_1$, and a similar idea in fact yields a map from $\mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$ to $\mathrm{Pic}(L\mathbb{P}_1)$ as follows. Let $p : Q \to \mathbb{P}_1$ be the principal $\mathbb{C}^*$-bundle associated with the hyperplane bundle $H \to \mathbb{P}_1$. Applying the loop functor we obtain a principal $L\mathbb{C}^*$-bundle $Lp : LQ \to L\mathbb{P}_1$. Now a homomorphism $\varphi : L\mathbb{C}^* \to \mathbb{C}^*$ determines a representation of $L\mathbb{C}^*$ on $\mathbb{C}$. Recall that, in general, with a principal G-bundle $P \to B$ and a representation ρ of G on a vector space V, one can functorially associate a vector bundle $E \to B$ with typical fiber V (see Section 12.5 of [4]). If h_{ab} are the G-valued transition functions of P with respect to some trivialization, the corresponding transition functions of E will be $\rho(h_{ab})$. Accordingly we associate with Lp and φ a line bundle Λ_φ. Define the map

$$\mathcal{L} : \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \to \mathrm{Pic}(L\mathbb{P}_1),\ \varphi \mapsto \Lambda_\varphi.$$

Note that the $PGL(2,\mathbb{C})$ action on $\mathbb{P}_1$ can be covered by a $GL(2,\mathbb{C})$ action on Q. One way to see this is to pass to the tautological $\mathbb{C}^*$-bundle Q^{-1}, whose total space is $\mathbb{C}^2 \setminus \{0\}$, on which the $GL(2,\mathbb{C})$ action is standard.

The $GL(2,\mathbb{C})$ action on Q gives rise to an $LGL(2,\mathbb{C})$ action on LQ. Since $LGL(2,\mathbb{C}) \to LPGL(2,\mathbb{C})$ is surjective (as follows from the exact homotopy sequence associated with the fibration $\mathbb{C}^* \to GL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$ and the lifting of homotopies), this $LGL(2,\mathbb{C})$ action will in fact cover the $LPGL(2,\mathbb{C})$ action on $L\mathbb{P}_1$. In particular, $\gamma^* LQ \cong LQ$ for $\gamma \in LPGL(2,\mathbb{C})$. Hence $\gamma^*\Lambda_\varphi \cong \Lambda_\varphi$ and we have proved the following:

Proposition 2.1. *The range of $\mathcal{L}$ is in* $\mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$.

Let $\mathfrak{U} = \{U_a = \mathbb{P}_1 \setminus \{a\} : a \in \mathbb{P}_1\}$. Then

$$L\mathfrak{U} = \{LU_a : a \in \mathbb{P}_1\} \tag{2.1}$$

is an open covering of $L\mathbb{P}_1$. Now we introduce a way to construct Čech cohomology classes in $H^1(L\mathfrak{U}, \mathcal{O}^*)$. Let $\mathcal{O}^G$ denote the sheaf of holomorphic maps to the complex Lie group G from a complex manifold. So $\mathcal{O}^{\mathbb{C}^*} = \mathcal{O}^*$. If $c = (c_{ab})$ is a Čech 1-cocycle of $\mathfrak{U}$ with values in the sheaf $\mathcal{O}^*$ and $[c]$ its cohomology class, then $Lc = (Lc_{ab})$ is a Čech 1-cocycle of $L\mathfrak{U}$ with values in $\mathcal{O}^{L\mathbb{C}^*}$. Any $\varphi \in \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$ induces a sheaf homomorphism $\mathcal{O}^{L\mathbb{C}^*} \to \mathcal{O}^*$. Since the cohomology class of $\varphi \circ Lc$ depends only on $[c]$, we obtain a map

$$H^1(\mathfrak{U}, \mathcal{O}^*) \times \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \to H^1(L\mathfrak{U}, \mathcal{O}^*), \ ([c], \varphi) \mapsto [\varphi \circ Lc], \tag{2.2}$$

a group homomorphism in both variables.

Fix $[c]$ in (2.2) to be the class $[c_H]$ of the hyperplane bundle $H \to \mathbb{P}_1$, where

$$\begin{aligned} c_H &= \{g_{ab} \in \mathcal{O}^*(U_a \cap U_b) : a, b \in \mathbb{P}_1,\ a \neq b\}, \\ g_{ab} &= \begin{cases} \frac{z-b}{z-a} & a, b \neq \infty \\ z - b & a = \infty \\ \frac{1}{z-a} & b = \infty \end{cases}, \ z \in U_a \cap U_b. \end{aligned} \tag{2.3}$$

Then we obtain a homomorphism $\mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \to H^1(L\mathfrak{U}, \mathcal{O}^*)$, $\varphi \mapsto [c_\varphi]$, where

$$c_\varphi = \{\varphi \circ Lg_{ab} \in \mathcal{O}^*(LU_a \cap LU_b) :\ a, b \in \mathbb{P}_1,\ a \neq b\}, \tag{2.4}$$

which has the same range as the map in (2.2).

Proposition 2.2. *The line bundle associated to $[c_\varphi]$ is Λ_φ. In particular $\mathcal{L}$ is a homomorphism.*

Proof. From the definitions it follows that $Lc_H = (Lg_{ab})$ is a family of transition functions of Lp, hence c_φ in (2.4) is a family of transition functions of Λ_φ. q.e.d.

Let $t \in S^1$ and $\Lambda_t \in \mathrm{Pic}(L\mathbb{P}_1)$ be the pull back of the hyperplane bundle on $\mathbb{P}_1$ by the evaluation map $E_t^{L\mathbb{P}_1}$. Obviously $E_t^{L\mathbb{C}^*} \in \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$, and by Proposition 2.2 $\mathcal{L}(E_t^{L\mathbb{C}^*}) = \Lambda_t$.

To prove Theorem 1.1 we need the following preparations:

Identify LU_∞ with $L\mathbb{C}$ and let $y \in L\mathbb{C}^*$. Define the map $\Phi_y : L\mathbb{C} \times \mathbb{P}_1 \ni (x, \lambda) \mapsto x + \lambda y \in L\mathbb{P}_1$, where by $\Phi_y(x, \infty)$ we mean the point loop $\infty \in L\mathbb{P}_1$. Since a continuous map $h : \Omega_1 \to \Omega_2$ between open subsets of Fréchet spaces is holomorphic if and only if its restriction to the intersection of Ω_1 with any affine line is holomorphic, see Sections 2.3, 3.1 of [2], one can easily check that Φ_y is holomorphic. Clearly $\Phi_y(x, \cdot) : \mathbb{P}_1 \to L\mathbb{P}_1$ is a holomorphic embedding, whose image we will denote by $v(x, y)$. The existence of subvarieties $v(x, y)$ immediately implies that any holomorphic function on $L\mathbb{P}_1$ is constant.

Let $\Lambda \in \mathrm{Pic}(L\mathbb{P}_1)$. If $\Lambda|_{\mathbb{P}_1} \cong H^n$, then we say that Λ is of order n, or $\mathrm{ord}(\Lambda) = n$; and we claim that $\Lambda|_{v(x,y)} \cong H^n$, $x \in L\mathbb{C}$, $y \in L\mathbb{C}^*$. This would imply that the only holomorphic section of Λ is the zero section if $\mathrm{ord}(\Lambda) < 0$. To show the claim, let c_1 denote the rational first Chern class of a line bundle. According to the first theorem of [10], the inclusion $\mathbb{P}_1 \to L\mathbb{P}_1$ induces an isomorphism $H^2(L\mathbb{P}_1, \mathbb{Q}) \cong H^2(\mathbb{P}_1, \mathbb{Q})$, so that $c_1(\Lambda)$ is completely determined by $c_1(\Lambda|_{\mathbb{P}_1}) = n$. Hence $c_1(\Lambda|_{v(x,y)})$, and therefore the degree of $\Lambda|_{v(x,y)}$, is also determined by n. That this degree is itself n then follows from computing it in the special case $\Lambda = \Lambda_t^n$.

Let $\varphi \in \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$. The restriction of φ to the subgroup of point loops must be of the type $z \mapsto z^n$, $z \in \mathbb{C}^*$, where $n \in \mathbb{Z}$. We call this n the order of φ and denote it by $\mathrm{ord}(\varphi)$. Proposition 2.2 implies that $\mathrm{ord}(\Lambda_\varphi) = \mathrm{ord}(\varphi)$. Let F_x be the fiber of Λ at $x \in L\mathbb{P}_1$.

Proposition 2.3. *Let $\Lambda \in \mathrm{Pic}(L\mathbb{P}_1)$, $\mathrm{ord}(\Lambda) = 0$ and $y \in L\mathbb{C}^*$. For any $\zeta \in F_\infty \setminus \{0\}$ there exists a unique non-vanishing section $\sigma = \sigma_{y,\zeta} \in H^0(LU_\infty, \Lambda)$ such that $\lim_{\lambda \to \infty} \sigma(x + \lambda y) = \zeta$, $x \in LU_\infty$. In particular, $\Lambda|_{LU_\infty}$ is holomorphically trivial.*

Proof. Since $\Lambda|_{v(x,y)}$ is trivial, uniqueness is obvious. As to existence, let $\mathcal{H}$ be a hyperplane in $L\mathbb{C}$ such that $y \notin \mathcal{H}$ and consider the line bundle $\widetilde{\Lambda} = \Phi_y^* \Lambda|_{\mathcal{H} \times \mathbb{P}_1}$. Since $\Phi_y \equiv \infty$ on $\mathcal{H} \times \{\infty\}$, $s = \Phi_y^* \zeta$ is a non-vanishing holomorphic section of $\widetilde{\Lambda}|_{\mathcal{H} \times \{\infty\}}$. In turn s determines a section $\widetilde{\sigma}$ of $\widetilde{\Lambda}|_{\mathcal{H} \times \mathbb{P}_1}$ such that $\widetilde{\sigma}|_{\{x\} \times \mathbb{P}_1}$ is the unique non-vanishing holomorphic section of $\widetilde{\Lambda}|_{\{x\} \times \mathbb{P}_1}$ with $\widetilde{\sigma}(x, \infty) = s(x, \infty)$, for all $x \in \mathcal{H}$. Next we show that $\widetilde{\sigma}$ is holomorphic. Let $x_0 \in \mathcal{H}$. By Proposition 5.1 of [5] there exist a neighborhood $x_0 \in U \subset \mathcal{H}$ and a section $v \in C^\infty(U \times \mathbb{P}_1, \widetilde{\Lambda})$ such that v is holomorphic on $\{x\} \times \mathbb{P}_1$ for all $x \in U$, and $v|_{\{x_0\}\times\mathbb{P}_1} = \widetilde{\sigma}|_{\{x_0\}\times\mathbb{P}_1} \neq 0$. By choosing a sufficiently small U we can assume $v \neq 0$. The function $\widetilde{\sigma}/v$ is C^∞ on $U \times \{\infty\}$ and is constant on $\{x\} \times \mathbb{P}_1$, $x \in U$, hence is C^∞ on $U \times \mathbb{P}_1$. So it follows that $\widetilde{\sigma} \in C^\infty(\mathcal{H} \times \mathbb{P}_1)$. Since $\bar{\partial}\widetilde{\sigma}|_{\mathcal{H}\times\{\infty\}} = 0$ and $\bar{\partial}\widetilde{\sigma}|_{\{x\}\times\mathbb{P}_1} = 0$ for all $x \in \mathcal{H}$, by Proposition 5.2(ii) of [5] we obtain that indeed $\bar{\partial}\widetilde{\sigma} = 0$. Then the desired σ is the pull back of $\widetilde{\sigma}$ by $(\Phi_y|_{\mathcal{H}\times\mathbb{C}})^{-1}$. q.e.d.

Since $H|_{U_\infty}$ is trivial, so is $\Lambda_t|_{LU_\infty}$. In general, let $\Lambda \in \mathrm{Pic}(L\mathbb{P}_1)$, $\mathrm{ord}(\Lambda) = n$. As

$$\Lambda = \Lambda_t^n \otimes (\Lambda_t^{-n} \otimes \Lambda), \text{ where } \mathrm{ord}(\Lambda_t^{-n} \otimes \Lambda) = 0, \tag{2.5}$$

Proposition 2.3 implies that $\Lambda|_{LU_\infty}$ is also trivial. More generally, $\Lambda|_{LU_a}$ is trivial, $a \in \mathbb{P}_1$, which means

Corollary 2.4. $\mathrm{Pic}(L\mathbb{P}_1) \cong H^1(L\mathfrak{U}, \mathcal{O}^*)$.

If $\Lambda \in \mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$, then Proposition 2.3 can be improved: σ there is essentially independent of y, and so is a canonical section of $\Lambda|_{LU_\infty}$.

Proposition 2.5. *Let $\Lambda \in \mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$, $\mathrm{ord}(\Lambda) \geq 0$. Then there exists a non-vanishing section $\sigma_\infty \in H^0(LU_\infty, \Lambda)$ such that $\lim_{\lambda\to\infty} \sigma_\infty(x + \lambda y)$ exists for all $x \in LU_\infty$ and $y \in L\mathbb{C}^*$. Such a section is unique up to a multiplicative constant.*

Proof. Suppose σ_∞ exists. For any $y \in L\mathbb{C}^*$, $\Lambda|_{v(0,y)} \cong H^{\mathrm{ord}(\Lambda)}$ has a unique holomorphic section which does not vanish on $v(0,y)\setminus\{\infty\}$ and assumes $\sigma_\infty(0)$ at 0; clearly σ_∞ agrees with this section on $v(0,y)\setminus\{\infty\}$. In particular, $\sigma_\infty(y)$ is uniquely determined by $\sigma_\infty(0)$, $y \in L\mathbb{C}^*$. Since $L\mathbb{C}^*$ is dense in $L\mathbb{C} = LU_\infty$, σ_∞ is completely determined by its value at 0, or unique up to a multiplicative constant. Next we show the existence.

First assume $\mathrm{ord}(\Lambda) = 0$. Let $\zeta \in F_\infty\setminus\{0\}$, $y_1, y_2 \in L\mathbb{C}^*$, and $\sigma_{y_1,\zeta}, \sigma_{y_2,\zeta} \in H^0(LU_\infty, \Lambda)$ as in Proposition 2.3. Define

$$h = h_{y_1,y_2} = \frac{\sigma_{y_1,\zeta}}{\sigma_{y_2,\zeta}} \in \mathcal{O}^*(LU_\infty),$$

which is independent of the choice of $\zeta \in F_\infty\setminus\{0\}$. For fixed $x_1 \in L\mathbb{C}$ let $\mu \in LPGL(2,\mathbb{C})$ be the translation $x \mapsto x + x_1$, $x \in L\mathbb{P}_1$. Clearly $\mu^*\sigma_{y_i,\zeta} \in H^0(LU_\infty, \mu^*\Lambda)$, $i = 1, 2$, is a section of the type as in Proposition 2.3. With a fixed isomorphism $\mu^*\Lambda \cong \Lambda$, $\mu^*\sigma_{y_i,\zeta}$ corresponds to $\sigma_{y_i,\zeta'}$, where

$\zeta' \in F_\infty \setminus \{0\}$. Therefore

$$h(x + x_1) = \mu^* h(x) = \frac{\mu^* \sigma_{y_1,\zeta}}{\mu^* \sigma_{y_2,\zeta}}(x) = \frac{\sigma_{y_1,\zeta'}}{\sigma_{y_2,\zeta'}}(x) = h(x).$$

Thus $h = h_{y_1,y_2}$ is a nonzero constant. From the definition of $\sigma_{y_2,\zeta}$ it follows that $\sigma_{y_1,\zeta} = h\sigma_{y_2,\zeta}$ extends to be a section of $\Lambda|_{v(x,y_2)}$ for all $x \in L\mathbb{C}$, where $y_2 \in L\mathbb{C}^*$ is arbitrary. The upshot is $\sigma_\infty = \sigma_{y_1,\zeta}$ with arbitrary y_1, ζ will do.

Second assume $\Lambda = \Lambda_t^n$, $n \geq 0$. If s is a section of H^n, nonzero on $\mathbb{P}_1 \setminus \{\infty\}$, take $\sigma_\infty = (E_t^{L\mathbb{P}_1})^* s$. Finally, these two special cases and (2.5) imply the general case. q.e.d.

Theorem 1.1 follows from:

Theorem 2.6. *$\mathcal{L}$: $\mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \to \mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$ is a group isomorphism.*

Proof. i) Injectivity: If $\mathcal{L}(\varphi) = \Lambda_\varphi$ is trivial, then $\mathrm{ord}(\varphi) = 0$; and by Proposition 2.2 and (2.4) we can find $f_a \in \mathcal{O}^*(LU_a)$, $a \in \mathbb{P}_1$, such that $f_a = (\varphi \circ Lg_{ab}) f_b$ on $LU_a \cap LU_b$. In particular, $f_\infty = \varphi f_0$ on $L\mathbb{C}^* \subset L\mathbb{P}_1$. For any $y \in L\mathbb{C}^*$ we have

$$\lim_{\mathbb{C} \ni \lambda \to \infty} f_\infty(\lambda y) = \lim_{\lambda \to \infty} \varphi(\lambda y) f_0(\lambda y) = \varphi(y) f_0(\infty),$$

since $\varphi(\lambda y) = \varphi(\lambda)\varphi(y) = \varphi(y)$. Thus $f_\infty|_{v(0,y)\setminus\{\infty\}}$ extends to all of $v(0,y)$, and so must be constant. In particular, $f_\infty(y) = f_\infty(0)$ and f_∞ itself is a constant. Similarly f_0 is also a constant. Then φ is a constant which can only be 1.

ii) Surjectivity: Since Λ_t^n is in the range of $\mathcal{L}$, by (2.5) we only need to show that any $\Lambda \in \mathrm{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$ with $\mathrm{ord}(\Lambda) = 0$ is in the range of $\mathcal{L}$.

Let $\varepsilon(x) = 1/x$, $x \in L\mathbb{P}_1$. The induced bundle $\varepsilon^*\Lambda$ is isomorphic to Λ, so Proposition 2.5 applies to produce a non-vanishing $\widetilde{\sigma}_\infty \in H^0(LU_\infty, \varepsilon^*\Lambda)$. Then $\sigma_0 = \widetilde{\sigma}_\infty \circ \varepsilon \in H^0(LU_0, \Lambda)$ is characterized, up to a multiplicative constant, by the fact that $\lim_{\lambda \to \infty} \sigma_0\left((x + \lambda y)^{-1}\right)$ exists, for all $x \in LU_\infty$, $y \in L\mathbb{C}^*$. To get rid of the ambiguity in the choice of the constant, fix a nonzero $s \in H^0(\mathbb{P}_1, \Lambda)$ and choose σ_0 and σ_∞ (as in Proposition 2.5) to agree with s on point loops. Set $\phi = \sigma_0/\sigma_\infty : L\mathbb{C}^* \to \mathbb{C}^*$. Note that on point loops $\phi = 1$. We will show that ϕ is a homomorphism and $\Lambda = \Lambda_\phi$.

For this purpose fix $y_1 \in L\mathbb{C}^*$ and define $\gamma(x) = y_1 x$, $x \in L\mathbb{P}_1$. It is straightforward that the non-vanishing section $\gamma^*\sigma_\infty \in H^0(LU_\infty, \gamma^*\Lambda)$ satisfies the conditions in Proposition 2.5. Hence under an isomorphism

$\gamma^*\Lambda \cong \Lambda$, $\gamma^*\sigma_\infty$ corresponds to a constant multiple of σ_∞. Similarly, under this isomorphism $\gamma^*\sigma_0$ corresponds to a constant multiple of σ_0. Therefore

$$\phi(y_1 y) = (\gamma^*\phi)(y) = \frac{\gamma^*\sigma_0}{\gamma^*\sigma_\infty}(y) = c\frac{\sigma_0}{\sigma_\infty}(y) = c\phi(y).$$

Letting $y = 1$ we get $c = \phi(y_1)$, so ϕ is indeed a homomorphism.

Finally with $a \in \mathbb{C}$ let $\mu_a(x) = x + a, x \in L\mathbb{P}_1$. For the bundle $\Lambda' = \mu_a^*\Lambda$ one can construct corresponding sections σ'_∞ and σ'_0; for the normalization, use $s' = \mu_a^* s \in H^0(\mathbb{P}_1, \Lambda')$. As $\Lambda' \cong \Lambda$, $\sigma'_0/\sigma'_\infty = \phi$. Since $(\mu_a^{-1})^*\sigma'_\infty \in H^0(LU_\infty, \Lambda)$ satisfies the conditions in Proposition 2.5 and agrees with s on point loops, $(\mu_a^{-1})^*\sigma'_\infty = \sigma_\infty$. Let $\sigma_a = (\mu_a^{-1})^*\sigma'_0 \in H^0(LU_a, \Lambda)$. Since

$$\frac{\sigma_a}{\sigma_\infty} = \left(\mu_a^{-1}\right)^* \frac{\sigma'_0}{\sigma'_\infty} = \phi \circ \mu_a^{-1},$$

it is straightforward to check that the transition functions $\sigma_b/\sigma_a \in \mathcal{O}^*(LU_a \cap LU_b)$ of Λ agree with those of Λ_ϕ given in (2.4); therefore $\Lambda = \Lambda_\phi$ is indeed in the range of $\mathcal{L}$. q.e.d.

Let $(L\mathbb{C})^*$ be the space of continuous linear functionals on $L\mathbb{C}$, let $\widetilde{\varphi} \in (L\mathbb{C})^*$ be the Lie algebra homomorphism induced by $\varphi \in \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$, and $x_0 \in L\mathbb{C}^*$ be a fixed loop whose winding number with respect to 0 is 1. The reader can check that the map

$$\mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*) \to \{\phi \in (L\mathbb{C})^* : \phi(1) \in \mathbb{Z}\} \times \mathbb{C}^*,\ \varphi \mapsto (\widetilde{\varphi}, \varphi(x_0))$$

is an isomorphism of groups. Therefore $\mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$ is a $\mathbb{Z}$-module of infinite rank.

The proof of Theorem 2.6 implies the following:

Proposition 2.7. *Let $\varphi \in \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$, $\mathrm{ord}(\varphi) \geq 0$. There is a family of non-vanishing sections $\{\sigma_a \in H^0(LU_a, \Lambda_\varphi) : a \in \mathbb{P}_1\}$, unique up to an overall multiplicative constant, that satisfies $\sigma_b/\sigma_a = \varphi \circ Lg_{ab}$ on $LU_a \cap LU_b$. Furthermore, σ_∞ satisfies the conditions in Proposition 2.5.*

Proof. Uniqueness is obvious: if $\{\sigma'_a\}$ is another family then $\{\sigma'_a/\sigma_a\}$ defines a holomorphic function on $L\mathbb{P}_1$, which, as we have said, must be a constant. When $\mathrm{ord}(\Lambda) = 0$, the family $\{\sigma_a\}$ is constructed in the proof of Theorem 2.6. When $\Lambda = \Lambda_t^n$, take sections τ_a of the hyperplane bundle $H \to \mathbb{P}_1$ such that $\tau_b/\tau_a = g_{ab}$ (see (2.3)), then $\sigma_a = (E_t^{L\mathbb{P}_1})^*\tau_a^n|_{LU_a}$ will do. The case of a general Λ now follows from (2.5). q.e.d.

3. Proof of Theorem 1.2(2)

We start with two results concerning polynomials on $L\mathbb{C}$. For simplicity let E_t denote both $E_t^{L\mathbb{C}} \in (L\mathbb{C})^*$ and $E_t^{L\mathbb{C}^*} \in \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$ till the end of this paper.

Lemma 3.1. *If a homogeneous polynomial $h \in \mathcal{O}(L\mathbb{C})$ of degree $n \geq 1$ does not vanish on $L\mathbb{C}^*$, then $h = cE_{t_1} \cdots E_{t_n}$, where $t_1, \ldots, t_n \in S^1$, and $c \neq 0$ is a constant.*

Proof. Since $L_\infty\mathbb{C}$ is embedded into $L\mathbb{C}$ with a dense image, we only need to show the lemma for the case of C^∞ loops.

Let $W = L\mathbb{C} \setminus L\mathbb{C}^*$, $\mathcal{Z}_h \subset W$ be the zero locus of h, and $\mathrm{Emb}(S^1, \mathbb{C}) \subset L_\infty\mathbb{C}$ be the open subset of embedded loops. Then $\mathcal{Z}_h \cap \mathrm{Emb}(S^1, \mathbb{C}) \neq \emptyset$. Otherwise for any $x \in \mathrm{Emb}(S^1, \mathbb{C})$, the polynomial $h(x+\lambda)$ in $\lambda \in \mathbb{C}$ has no zero, hence is constant. In particular, $h(x+1) = h(x)$ on $\mathrm{Emb}(S^1, \mathbb{C})$ and therefore on $L_\infty\mathbb{C}$. So $h(1) = h(0) = 0$. But $1 \in L\mathbb{C}^*$, contradiction.

Let $x_1 \in W \cap \mathrm{Emb}(S^1, \mathbb{C})$. Next we show that W is a submanifold of real codimension one near x_1. Let t_1 be the unique element of $S^1 = \mathbb{R}/\mathbb{Z}$ such that $x_1(t_1) = 0$. We can assume that $x_1'(t_1)$ is not real, otherwise replace x_1 by ix_1. Consider the equation

$$\operatorname{Im} x(s) = 0, \; x \in L_\infty\mathbb{C}, \, s \in S^1. \tag{3.1}$$

This equation can also be considered on the C^k loop space $L_k\mathbb{C}$, $1 \leq k < \infty$. Note that $\operatorname{Im} x_1'(t_1) \neq 0$. Apply the Implicit Function Theorem on Banach spaces (see Theorem 2.5.7 of [1]) to the C^k map $L_k\mathbb{C} \times S^1 \to \mathbb{R}$, $(x,t) \mapsto \operatorname{Im} x(t)$ near (x_1, t_1). When $k = 1$ we obtain a neighborhood $U_1 \subset L_1\mathbb{C}$ of x_1 consisting of embedded loops, a neighborhood $V \subset S^1$ of t_1, and a C^1 map $\phi : U \to V$ such that for any $x \in U$, $s = \phi(x)$ is the unique solution of (3.1) in V. We can shrink U_1 and V if necessary to ensure that $\operatorname{Im} x'(t) \neq 0$ for all $(x,t) \in U_1 \times V$. For arbitrary $k < \infty$ we obtain that ϕ is C^k on $U_k = U_1 \cap L_k\mathbb{C}$ by the Implicit Function Theorem at any $(x, \phi(x))$, $x \in U_k$. This implies that ϕ is C^∞ on $U = U_1 \cap L_\infty\mathbb{C}$. We can choose sufficiently small U so that $x(t) \neq 0$ if $x \in U$ and $t \notin V$; then

$$x\left(\phi(x)\right) = 0, \; x \in U \cap W. \tag{3.2}$$

Let Y be the real hyperplane $\{y \in L_\infty\mathbb{C} : \operatorname{Im} y(t_1) = 0\}$ of $L_\infty\mathbb{C}$. Define the C^∞ map $\tau : S^1 \times Y \to L_\infty\mathbb{C}$, $(s, x(t)) \mapsto x(t-s)$. The C^∞ map $\rho : U \to S^1 \times Y$, $x(t) \mapsto (\phi(x) - t_1, x(t + \phi(x) - t_1))$ is the local inverse of τ near (t_1, x_1). Let $\mathcal{H}_t$ be the kernel of E_t. Note that $\mathcal{H}_{t_1} \subset Y$. Since

$\tau(S^1\times\mathcal{H}_{t_1}) = W$ and by (3.2) $\rho(W\cap U)\subset S^1\times\mathcal{H}_{t_1}$, $W\cap U$ is a submanifold of real codimension one of U. Its tangent space is

$$TW_x = \mathcal{H}_{\phi(x)} \oplus \{x'\mathbb{R}\},\ x\in W\cap U, \tag{3.3}$$

if we identify the tangent space of $L_\infty\mathbb{C}$ at x with $L_\infty\mathbb{C}$.

Now assume $x_1\in \mathrm{Emb}(S^1,\mathbb{C})\cap\mathcal{Z}_h$. Let $v\in\mathcal{H}_{t_1}$ and $v_0\notin W$. Consider the restriction of h to the 2-dimensional affine subspace $E=\{x_1+\lambda_1 v+\lambda_2 v_0 : \lambda_1,\lambda_2\in\mathbb{C}\}$. Let $\delta:\Delta\to\mathcal{Z}_h\cap E\cap U\subset W$ be a holomorphic arc such that $\delta(0)=x_1$ and $0\neq\delta'(0)\in TW_{x_1}$. As for all $\lambda\in\Delta$ $\delta'(\lambda)$ is in the maximal complex subspace of $T_{\delta(\lambda)}W$, $\delta'(\lambda)\in\mathcal{H}_{\phi(\delta(\lambda))}$ by (3.3). Since ϕ is constant on $\mathcal{H}_{\phi(\delta(\lambda))}\cap U$, the derivative of ϕ in the direction of $\delta'(\lambda)$ is zero, hence $\phi\circ\delta\equiv t_1$. In view of (3.2) $\delta(\Delta)\subset\mathcal{H}_{t_1}\cap E$. It follows that $\{x_1+\lambda_1 v\}\subset\mathcal{Z}_h$ and $\mathcal{H}_{t_1}\subset\mathcal{Z}_h$.

The local ring $\mathcal{O}(L_\infty\mathbb{C})_x$, $x\in L_\infty\mathbb{C}$, is a unique factorization domain, see Proposition 5.15 of [7]. The germ of E_{t_1} at any $x\in\mathcal{H}_{t_1}$ is prime, for the functions $e(\lambda)=E_{t_1}(x+\lambda y)$ vanish to first order at $\lambda=0$ if $y(t_1)\neq 0$. Applying the Nullstellensatz in Theorem 5.14 of [7] to the prime ideals $((E_{t_1})_x)$, $x\in\mathcal{H}_{t_1}$, we obtain that $h=E_{t_1}\widetilde{h}$, $\widetilde{h}\in\mathcal{O}(L_\infty\mathbb{C})$. On any affine line $\widetilde{h}$ is a polynomial of degree $\leq n-1$, so it is a polynomial of degree $\leq n-1$ on $L_\infty\mathbb{C}$, see Section 2.2 of [2]. The zero locus of $\widetilde{h}$ is still in W, so that repeating the above process we obtain the conclusion of the lemma.

q.e.d.

Let $\mathcal{P}^n(L\mathbb{C})$ be the space of holomorphic polynomials of degree $\leq n$ on $L\mathbb{C}$.

Proposition 3.2. *Let $h_1,h_2\in\mathcal{O}(L\mathbb{C})$ and $\varphi\in\mathcal{O}(L\mathbb{C}^*)$ such that $\varphi(\lambda y)=\lambda^n\varphi(y)$, $\lambda\in\mathbb{C}^*$, $y\in L\mathbb{C}^*$, $n\in\mathbb{N}\cup\{0\}$. If $h_1(y)=\varphi(y)h_2(y^{-1})$, $y\in L\mathbb{C}^*$, then $h_1,h_2\in\mathcal{P}^n(L\mathbb{C})$. Let h_j^i be the i-th order homogeneous component of h_j, $j=1,2$, $i=0,\dots,n$. Then $h_1^i(y)=\varphi(y)h_2^{n-i}(y^{-1})$.*

Proof. If $x\in L\mathbb{C}$, $y\in L\mathbb{C}^*$, then

$$\lim_{\lambda\to\infty}\frac{h_1(x+\lambda y)}{\lambda^n}=\lim_{\lambda\to\infty}\varphi(x\lambda^{-1}+y)h_2((x+\lambda y)^{-1})=\varphi(y)h_2(0).$$

Thus $h_1(x+\lambda y)$ is a polynomial of degree $\leq n$ in λ. As $L\mathbb{C}^*$ is dense in $L\mathbb{C}$, the same holds for all $y\in L\mathbb{C}$. Since a continuous function on a Fréchet space is a polynomial of degree $\leq n$ if its restriction to any affine line is such a polynomial, see Section 2.2 of [2], we conclude that $h_1\in\mathcal{P}^n(L\mathbb{C})$. Similarly $h_2\in\mathcal{P}^n(L\mathbb{C})$. Comparing homogeneous components of same order on both sides of the equation $h_1(y) = \varphi(y)h_2(y^{-1})$, we get $h_1^i(y) = \varphi(y)h_2^{n-i}(y^{-1})$, $i = 0, \dots, n$. q.e.d.

Let $\sigma \in H^0(L\mathbb{P}_1, \Lambda_\varphi)$, $\operatorname{ord}(\varphi) \geq 0$. With sections $\sigma_a \in H^0(LU_a, \Lambda_\varphi)$, $a \in \mathbb{P}_1$, in Proposition 2.7, the functions $H_a = \sigma/\sigma_a \in \mathcal{O}(LU_a)$ satisfy

$$H_a(x) = \varphi \circ Lg_{ab}(x)H_b(x),\ x \in LU_a \cap LU_b. \tag{3.4}$$

Note that Lg_{ab} maps LU_a to $L\mathbb{C}$ biholomorphically, and let $h_{ab} = H_a \circ Lg_{ab}^{-1} \in \mathcal{O}(L\mathbb{C})$. Then (3.4) implies that

$$h_{ab}(y) = \varphi(y)h_{ba}(y^{-1}),\ y \in L\mathbb{C}^*. \tag{3.5}$$

From Proposition 3.2 it follows that $h_{ab} \in \mathcal{P}^n(L\mathbb{C})$; also

$$H_\infty = h_{\infty 0} \in \mathcal{P}^n(L\mathbb{C}). \tag{3.6}$$

Proof of Theorem 1.2(2).

By Theorem 2.6 each element of $\operatorname{Pic}(L\mathbb{P}_1)^{LPGL(2,\mathbb{C})}$ is of the type Λ_φ, $\varphi \in \operatorname{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$. Suppose Λ_φ has a nonzero holomorphic section σ. We can assume that $\sigma(\infty) \neq 0$; this can be arranged by pulling σ back by a suitable element of $LPGL(2, \mathbb{C})$. Then $\Lambda_\varphi|_{\mathbb{P}_1} \cong H^n$ also has a nonzero section, so $\operatorname{ord}(\varphi) = n \geq 0$. Note that $h_{0\infty}(0) = H_0(\infty) \neq 0$. By (3.5), applied with $ab = \infty 0$, and Proposition 3.2

$$h_{\infty 0}^n(y) = \varphi(y)h_{0\infty}(0) \neq 0,\ y \in L\mathbb{C}^*. \tag{3.7}$$

Hence $h = h_{\infty 0}^n \in \mathcal{O}(L\mathbb{C})$ satisfies the condition in Lemma 3.1, so $h_{\infty 0}^n = cE_{t_1} \cdots E_{t_n}$. By (3.7) $\varphi = E_{t_1} \cdots E_{t_n}$, and Λ_φ is of the type as in Theorem 1.2(1). q.e.d.

4. The space of holomorphic sections

In this section we shall study the space $H^0(L\mathbb{P}_1, \Lambda)$, where Λ is as in Theorem 1.2(1), and we shall prove Theorem 1.2(1). Since $\dim H^0(L\mathbb{P}_1, \Lambda) = 1$ if Λ is trivial, we fix $\Lambda = \Lambda_\varphi$ nontrivial, where

$$\varphi = E_{t_1}^{n_1} \cdots E_{t_r}^{n_r},\ n_i > 0,\ t_i \neq t_j \text{ if } i \neq j. \tag{4.1}$$

With $\sigma_a \in H^0(LU_a, \Lambda_\varphi)$ as in Proposition 2.7 and $\sigma \in H^0(L\mathbb{P}_1, \Lambda_\varphi)$, we have seen that $\sigma/\sigma_\infty = H_\infty \in \mathcal{P}^n(L\mathbb{C})$, cf. (3.6). Define a monomorphism

$$\mathfrak{H}_\varphi : H^0(L\mathbb{P}_1, \Lambda_\varphi) \to \mathcal{P}^n(L\mathbb{C}),\ \sigma \mapsto \sigma/\sigma_\infty,$$

where $n = \operatorname{ord}(\varphi) \geq 0$. Let $\mathrm{R}(\mathfrak{H}_\varphi) \subset \mathcal{P}^n(L\mathbb{C})$ be the range of $\mathfrak{H}_\varphi$. We shall study $H^0(L\mathbb{P}_1, \Lambda_\varphi)$ through $\mathrm{R}(\mathfrak{H}_\varphi)$.

The bundle Λ_φ, where φ is as in (4.1), has nontrivial holomorphic sections: products of pull back sections by evaluation maps on $L\mathbb{P}_1$.

Proposition 4.1. *The linearly independent functions $E_{t_1}^{m_1} \cdots E_{t_r}^{m_r}$ are in $\mathrm{R}(\mathfrak{H}_\varphi)$, $0 \leq m_i \leq n_i, 1 \leq i \leq r$. In particular, $(n_1+1)\cdots(n_r+1) \leq \dim H^0(L\mathbb{P}_1, \Lambda_\varphi)$.*

Proof. Choose a basis $\{\widetilde{\tau}_j : j = 0, \ldots, n_1\}$ of $H^0(\mathbb{P}_1, H^{n_1})$, where $\widetilde{\tau}_0$ has a zero of order n_1 at $\infty \in \mathbb{P}_1$ and $\widetilde{\tau}_j = z^j \widetilde{\tau}_0$, $z \in \mathbb{P}_1$. Let $\tau_j = (E_{t_1}^{L\mathbb{P}_1})^* \widetilde{\tau}_j \in H^0(L\mathbb{P}_1, \Lambda_{t_1}^{n_1})$. The section $\tau_0|_{LU_\infty}$ satisfies the conditions in Proposition 2.5. Therefore

$$\mathfrak{H}_\varphi(\tau_j) = \tau_j/\tau_0 = E_{t_1}^* z^j = E_{t_1}^j.$$

Similarly we have pull back sections of $\Lambda_{t_i}^{n_i}$, $i = 1, \ldots, r$. By taking products of such sections we obtain sections of Λ_φ, hence $E_{t_1}^{m_1} \cdots E_{t_r}^{m_r} \in \mathrm{R}(\mathfrak{H}_\varphi), 0 \leq m_i \leq n_i, 1 \leq i \leq r$. Another way of obtaining these functions is to pull back monomials on $\mathbb{C}^r$ by the surjective map $(E_{t_1}, \ldots, E_{t_r}) : L\mathbb{C} \to \mathbb{C}^r$, which interpretation proves the claim of linear independence. q.e.d.

The elements of $\mathrm{R}(\mathfrak{H}_\varphi)$ identified in Proposition 4.1 are in the subalgebra of $\mathcal{P}^n(L\mathbb{C})$ generated by evaluation functions. In other words they are polynomials in finitely many linear functionals. The next proposition shows that this is true in general. For convenience, in Propositions 4.2 and 4.3 we shall restrict our discussion to C^k $(1 \leq k \leq \infty)$ loop spaces $L_k\mathbb{P}_1$. Let $x^{(\nu)}(t)$ denote the ν-th derivative of $x \in L_k\mathbb{C}$ at $t \in S^1 = \mathbb{R}/\mathbb{Z}$, $\nu \leq k$. Note that the function $x \to x^{(\nu)}(t)$ is in $(L_k\mathbb{C})^*$.

Proposition 4.2. *If $\sigma \in H^0(L_k\mathbb{P}_1, \Lambda_\varphi)$ and $P = \mathfrak{H}_\varphi(\sigma)$, then $P(x)$ is a polynomial in finitely many derivatives $x^{(\nu)}(t_i)$, $0 \leq \nu \leq k$, $1 \leq i \leq r$.*

Proof. Let $A = \{t_1, \ldots, t_r\} \subset S^1$, $x_0 \in L_k U_\infty$, and denote the k-jet of $x \in L_k\mathbb{P}_1$ by $j^k x$. Define

$$Z = Z(k, A, x_0) = \{x \in L_k\mathbb{P}_1 : j^k x|_A = j^k x_0|_A\}.$$

This is a connected complex submanifold of $L_k\mathbb{P}_1$ and any holomorphic function on it is a constant, see Sections 3, 4 of [6]. Consider the sections σ_a of Proposition 2.7. In the proof of that proposition we have shown that, when $\Lambda = \Lambda_t^n$, σ_a can be taken to be the pullback by $E_t^{L\mathbb{P}_1}$ of sections τ_a of the hyperplane bundle $H \to \mathbb{P}_1$, $\tau_a \neq 0$ on U_a. When $\Lambda = \otimes \Lambda_{t_i}^{n_i}$, σ_a can be taken as the product of such sections. Then $\sigma_a(x) = 0$ only if $x(t_i) = a$

for some i; in particular $\sigma_\infty \neq 0$ on Z. It follows that $\sigma/\sigma_\infty|_Z \in \mathcal{O}(Z)$ is constant, i.e., P is constant on any affine subspace

$$\{x \in L_k\mathbb{C} : j^k x|_A = j^k x_0|_A\},\ x_0 \in L_k\mathbb{C}. \tag{4.2}$$

When $k < \infty$, the continuous linear functionals $x \to x^{(\nu)}(t_i)$, $0 \leq \nu \leq k$, $1 \leq i \leq r$, give rise to a surjective linear map $J : L_k\mathbb{C} \to \mathbb{C}^{(k+1)r}$, whose fibers are the affine subspaces in (4.2). What we have shown above implies that $P = J^*f$, where f is a function on $\mathbb{C}^{(k+1)r}$. In fact f is a holomorphic polynomial of degree $\leq n$, for with a linear right inverse I to J we have $f = P \circ I$.

Consider the case when $k = \infty$. Let P^j be the j-th order homogeneous component of $P \in \mathcal{P}^n(L_\infty\mathbb{C})$, $j = 1, \ldots, n$. These components are also constant on any affine subspace in (4.2). By the definition of a homogeneous polynomial we can find a continuous symmetric j-linear mapping $\Psi_j : (L_\infty\mathbb{C})^j \to \mathbb{C}$ such that $P^j(x) = \Psi_j(x, \ldots, x)$. Applying the Schwartz Kernel Theorem (see Theorem 5.2.1 of [3]) one can show that there exists a distribution K_j in the j dimensional torus T^j such that

$$\Psi_j(x_1, \ldots, x_j) = K_j(x_1(s_1) \cdots x_j(s_j)),\ (s_1, \ldots, s_j) \in T^j.$$

So $P^j(x) = K_j(x(s_1) \cdots x(s_j))$. The Polarization Formula (see (2) in Section 2.2 of [2])

$$K_j\left(x_1(s_1) \cdots x_j(s_j)\right) = \frac{1}{2^j j!} \sum_{\varepsilon_1, \ldots, \varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_j P^j(\varepsilon_1 x_1 + \cdots + \varepsilon_j x_j)$$

and the fact that P^j depends only on $j^\infty x|_A$ imply that K_j is supported in $A^j \subset T^j$. Therefore K_j is a (finite) linear combination of partial derivatives at points in A^j, see Theorem 2.3.4 of [3]. Hence P^j and P are polynomials in finitely many $x^{(\nu)}(t_i)$, $x \in L_\infty\mathbb{C}$. q.e.d.

For each i let N_i be the order of the highest derivative $x^{(\nu)}(t_i)$ that $P(x)$ depends on, see Proposition 4.2, and m_i the degree of $P(x)$ as a polynomial of $x^{(N_i)}(t_i)$. Our next task is to estimate N_i, m_i.

Proposition 4.3. *$m_i\,(N_i + 1) \leq n_i$, where n_i is defined in* (4.1).

Proof. Fix i. At first assume $P = \mathfrak{H}_\varphi(\sigma)$ contains the monomial

$$c_1\, x^{(N_i)}(t_i)^{m_i}, \tag{4.3}$$

where $c_1 \neq 0$ is a constant. By (3.6) and (3.5) $P = h_{\infty 0}$ satisfies $h_{0\infty}(y) = \varphi(y)h_{\infty 0}(y^{-1})$ so that Proposition 3.2 implies that

$$h^{n-m_i}_{0\infty}(x) = \varphi(x)\left[P^{m_i}(x^{-1})\right],\ x \in L_k\mathbb{C}^*, \tag{4.4}$$

where superscripts indicate homogeneous components of the given order. Since P^{m_i} is a polynomial in $x^{(\nu)}(t_j)$, $P^{m_i}(x^{-1})$ is a sum of rational expressions, where the denominators are monomials in $x(t_j)$, $1 \leq j \leq r$, and the numerators are monomials in $x^{(\nu)}(t_j)$, $\nu \geq 1$. In this sum the monomial in (4.3) gives rise to the term

$$c_1(-1)^{m_i N_i}(N_i!)^{m_i}\frac{x^{(1)}(t_i)^{m_i N_i}}{x(t_i)^{m_i(N_i+1)}},$$

which is the only term in $P^{m_i}(x^{-1})$ with this high or higher power of $x(t_i)$ in the denominator. Hence taking

$$x(t) = x_s(t) = e^{2\pi i(t-t_i)} - 1 + s,\ s \in (-1, 0),$$

in (4.4) and noting that $\varphi(x_s) = \prod_j x_s(t_j)^{n_j}$, we obtain

$$h^{n-m_i}_{0\infty}(x_s) = s^{n_i - m_i(N_i+1)}\prod_{j\neq i} x_s(t_j)^{n_j}\left(c_2 + sg(s)\right), \tag{4.5}$$

where $c_2 \neq 0$ is a constant, $g(s)$ is a rational function in s which is bounded on the interval $(-1, 0)$. In order that the limit of right-hand side of (4.5) exist as $s \to 0^-$ we must have $m_i(N_i + 1) \leq n_i$.

The proof is finished if we can show that for any $\sigma \in H^0(L_k\mathbb{P}_1, \Lambda_\varphi)$ there exists $\sigma' \in H^0(L_k\mathbb{P}_1, \Lambda_\varphi)$ such that the corresponding $N'_i = N_i$, $m'_i = m_i$, and $\mathfrak{H}_\varphi(\sigma')$ contains the monomial (4.3).

We write P in the form

$$P(x) = x^{(N_i)}(t_i)^{m_i} f_1(x) + f_2(x),\ x \in L_k\mathbb{C},$$

where $f_1(x), f_2(x)$ are polynomials in $x^{(\nu)}(t_j)$, $f_1 \neq 0$ is independent of $x^{(N_i)}(t_i)$ and the degree of f_2 in $x^{(N_i)}(t_i)$ is strictly less than m_i. Let $y \in L_k\mathbb{C}^*$, and $\gamma \in L_k PSL(2, \mathbb{C})$ be the map $x \mapsto y(x + 1)$, $x \in L_k\mathbb{P}_1$. Let P' denote $\mathfrak{H}_\varphi(\gamma^*\sigma)$. From Propositions 2.7 and 2.5 we obtain that $\gamma^*\sigma_\infty = c_3\sigma_\infty$ (with a fixed isomorphism $\gamma^*\Lambda_\varphi \cong \Lambda_\varphi$), where $c_3 \neq 0$ is a constant. Therefore

$$P'(x) = c_3\frac{\gamma^*\sigma}{\gamma^*\sigma_\infty}(x) = c_3\gamma^* P(x) = c_3 P\left(y(x+1)\right),\ x \in L_k\mathbb{C}. \tag{4.6}$$

Computing the right-hand side of (4.6) by the product rule, we find that it contains the monomial

$$c_3\, y(t_i)^{m_i} f_1(y) \left(x^{(N_i)}(t_i)\right)^{m_i}.$$

Since $f_1 \neq 0$, we can find y such that $f_1(y) \neq 0$; then we can choose $\sigma' = \gamma^*\sigma$. q.e.d.

Proof of Theorem 1.2(1). The lower bound is given in Proposition 4.1. For the case of C^∞ loops Propositions 4.2 and 4.3 imply that $\mathrm{R}(\mathfrak{H}_\varphi))$ consists of polynomials of degree $\leq \mathrm{ord}(\Lambda_\varphi)$ in $x^{(\nu)}(t_j)$, $0 \leq \nu \leq n_j - 1$, $1 \leq j \leq r$, therefore $\dim H^0(L_\infty\mathbb{P}_1, \Lambda_\varphi) < \infty$. Since in general $L_\infty\mathbb{P}_1$ is continuously embedded into $L\mathbb{P}_1$ with a dense image, so that the restriction map $H^0(L\mathbb{P}_1, \Lambda_\varphi) \to H^0(L_\infty\mathbb{P}_1, \Lambda_\varphi)$ is monomorphic, we conclude that $\dim H^0(L\mathbb{P}_1, \Lambda_\varphi) < \infty$. q.e.d.

An immediate application of Proposition 4.3 is to identify all holomorphic sections of a "generic" bundle of the type considered in Theorem 1.2(1).

Corollary 4.4. *If* $n_1 = \cdots = n_r = 1$ *in* (4.1), *then* $\dim H^0(L\mathbb{P}_1, \Lambda_\varphi) = 2^r$.

Proof. Proposition 4.3 gives that $\mathrm{R}(\mathfrak{H}_\varphi)$ only contains polynomials in evaluation maps $E_{t_1}, \ldots, E_{t_r}$, of degree ≤ 1 in each variable, and all such polynomials are indeed in $\mathrm{R}(\mathfrak{H}_\varphi)$ by Proposition 4.1. q.e.d.

There are bundles as in Theorem 1.2(1) which have holomorphic sections other than those identified in Proposition 4.1. We shall show this by an explicit construction.

Proposition 4.5. *Let* $h \in \mathcal{O}(L\mathbb{C})$ *and* $\varphi \in \mathrm{Hom}(L\mathbb{C}^*, \mathbb{C}^*)$. *Then* $h \in \mathrm{R}(\mathfrak{H}_\varphi)$ *if and only if there exists a family* $\{f_a \in \mathcal{O}(L\mathbb{C}) : a \in \mathbb{C}\}$ *such that*

$$h(x+a) = \varphi(x) f_a(x^{-1}),\ x \in L\mathbb{C}^*. \tag{4.7}$$

Proof. With σ_a of Proposition 2.7 (4.7) is equivalent to

$$h(x)\sigma_\infty(x) = f_a\left((x-a)^{-1}\right)\sigma_a(x),\ x \in LU_a \cap LU_\infty,\ a \in \mathbb{C},$$

which defines a section $\sigma \in H^0(L\mathbb{P}_1, \Lambda_\varphi)$ such that $\mathfrak{H}_\varphi(\sigma) = h$ and vice versa. q.e.d.

From now on, until the last paragraph, we shall work with the space of C^k loops. Let $h \in \mathcal{O}(L_k\mathbb{C}^*)$ be a rational function of finitely many $x^{(\nu)}(t_j)$,

$0 \le \nu \le k$, $1 \le j \le r$. Letting each t_j vary in S^1, h induces a function $\chi \in C(L_k\mathbb{C}^* \times (S^1)^r)$. If h does not depend on the highest derivative $x^{(k)}(t_i)$ for some i (this is automatically satisfied if $k = \infty$), then χ is differentiable in t_i, and we define

$$h^{t_i}(x) = \frac{\partial}{\partial t_i}\chi(x, t_1, \dots, t_r).$$

Thus $h^{t_i} \in \mathcal{O}(L_k\mathbb{C}^*)$ is also a rational function in $x^{(\nu)}(t_j)$, $0 \le \nu \le k$, $1 \le j \le r$. If h was a polynomial, so will be h^{t_i}. The linear map $T^i : h \mapsto h^{t_i}$ satisfies $T^i(h_1h_2) = T^i(h_1)h_2 + h_1T^i(h_2)$, $T^i(h(x^{-1})) = (T^ih)(x^{-1})$, and $T^i(h(x+a)) = (T^ih)(x+a)$ if $h \in \mathcal{O}(L_k\mathbb{C})$.

Proposition 4.6. *Let $\varphi_i = \varphi E_{t_i}$, and assume that $h \in \mathrm{R}(\mathfrak{H}_\varphi)$ does not depend on $x^{(k)}(t_i)$ for some i. Then $h^{t_i} \in \mathrm{R}(\mathfrak{H}_{\varphi_i})$.*

Proof. If f_a is defined as in Proposition 4.5, Proposition 3.2 implies $f_a \in \mathcal{P}^n(L\mathbb{C})$. Since h and φ depend only on $x^{(\nu)}(t_j)$, so does f_a: it is in fact a polynomial in $x^{(\nu)}(t_j)$ ($\nu < k$ if $j = i$, $\nu \le k$ for all other j). Applying T^i to (4.7) we obtain

$$\begin{aligned} h^{t_i}(x+a) &= T^i\big(h(x+a)\big) \\ &= n_i\varphi(x)x(t_i)^{-1}x^{(1)}(t_i)f_a(x^{-1}) + \varphi(x)f_a^{t_i}(x^{-1}) \\ &= \varphi_i(x)\widetilde{f}_a(x^{-1}), \end{aligned}$$

where $\widetilde{f}_a(x) = -n_ix^{(1)}(t_i)f_a(x) + x(t_i)f_a^{t_i}(x)$, so that $h^{t_i} \in \mathrm{R}(\mathfrak{H}_{\varphi_i})$ by Proposition 4.5. q.e.d.

Finally we shall discuss the order N_i of the highest derivative $x^{(\nu)}(t_i)$ that $\mathfrak{H}_\varphi(\sigma)$ can depend on, in the case of the C^k loop space $L_k\mathbb{P}_1$. Recall from Propositions 4.2, 4.3 that $N_i \le \min(k, n_i - 1)$. It turns out that this estimate is sharp:

Theorem 4.7. *There exists a section of Λ_φ for which*

$$N_i = \min(k, n_i - 1).$$

Proof. With $\mu_i = \min(k, n_i - 1)$ and $0 \le \nu \le \mu_i$ let $\varphi_\nu = \varphi E_{t_i}^{\nu-\mu_i}$. Proposition 4.1 gives that $E_{t_i} \in \mathrm{R}(\mathfrak{H}_{\varphi_0})$. Repeatedly applying Proposition 4.6 we obtain that the functions $h_\nu(x) = x^{(\nu)}(t_i)$ are in $\mathrm{R}(\mathfrak{H}_{\varphi_\nu})$ for $\nu \le \mu_i$. Thus $h_{\mu_i}(x) = x^{(\mu_i)}(t_i)$ is in the range of $\mathfrak{H}_\varphi$, and $N_i = \mu_i$ if $\sigma = \mathfrak{H}_\varphi^{-1}h_{\mu_i}$. q.e.d.

A similar reasoning would apply to the space of $W^{k,p}$ loops, where the largest value for N_i turns out to be $\min(k-1, n_i-1)$. A remarkable consequence of this is that while for generic bundles Λ_φ, as we have seen in

Corollary 4.4, $\dim H^0(L\mathbb{P}_1, \Lambda_\varphi)$ does not vary with the regularity of loops, when at least one $n_i > 1$, this dimension will depend on the regularity class C^k, $W^{k,p}$ considered.

References

[1] R. Abraham, J.E. Marsden & T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd ed., Springer, New York, 1988, MR 0960687, Zbl 0875.58002.

[2] M. Hervé, *Analyticity in Infinite Dimensional Spaces*, Walter de Gruyter & Co., Berlin, 1989, MR 0986066, Zbl 0666.58008.

[3] L. Hörmander, *The Analysis of Linear Partial Differential Operators.* I. *Distribution Theory and Fourier Analysis*, Springer, Berlin, 1983, MR 0717035, Zbl 0521.35001.

[4] D. Husemoller, *Fibre Bundles*, 2nd ed., Graduate Texts in Mathematics, **20**, Springer, New York, 1975, MR 0370578, Zbl 0307.55015.

[5] L. Lempert, *The Dolbeault complex in infinite dimensions*, I, J. Amer. Math. Soc. **11** (1998) 485–520, MR 1603858, Zbl 0904.32014.

[6] L. Lempert, *Holomorphic functions on* (*generalized*) *loop spaces*, Proc. Royal Irish Acad., to appear.

[7] P Mazet, *Analytic sets in locally convex spaces*, North Holland, Amsterdam, 1984, MR 0756238, Zbl 0588.46032.

[8] J. McClure, *Private communication*.

[9] J.J. Millson & B. Zombro, *A Kähler structure on the moduli space of isometric maps of a circle into Euclidean space*, Invent. Math. **123** (1996) 35–59, MR 1376245, Zbl 0859.58007.

[10] M. Vigué-Poirrier & D. Sullivan, *The homology theory of the closed geodesic problem*, J. Differential Geom. **11** (1976) 633–644, MR 0455028, Zbl 0361.53058.

[11] E. Witten, *The index of the Dirac operator in loop space*, Elliptic curves and modular forms in algebraic topology, Lecture Notes in Math., **1326**, Springer, Berlin, 1988, 161–181, MR 0970288, Zbl 0679.58045.

Department of Mathematics
Purdue University
West Lafayette, IN 47906